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Manifolds and Group actions

Homework 4

Mandatory Exercise 1. (5 points)
Note that the set $M_n(\mathbb{R})$ of square n dimensional matrices can be identified with $\mathbb{R}^{n^2}$ and has a natural manifold structure. Consider the set of orthogonal matrices

$$O(n) := \{A \in M_n(\mathbb{R}) \,|\, AA^T = \mathrm{id}\}.$$

a) Show that that the set of orthogonal matrices is a submanifold of $M_n(\mathbb{R})$.

b) What is the dimension of $O(n)$?

c) Describe the tangent space of $O(n)$ at the identity.

Mandatory Exercise 2. (5 points)
Discuss and prove/disprove whether the following maps are injective and if they are immersions/embeddings. Are the images submanifolds?

a) Let $c \in \mathbb{R}$ and $\gamma : \mathbb{R} \to S^1 \times S^1$ be the map

$$\gamma(t) = (e^{2\pi i t}, e^{2\pi i c t}).$$

Hint: Be careful, your answer should depend on number theoretic properties of c.

b) Let $f : M \to N$ be any given smooth map. Consider the map $\mathrm{graph}_f : M \to M \times N$ defined by $\mathrm{graph}_f(x) = (x, f(x))$.

Mandatory Exercise 3. (5 points)
Show that the following definitions of a submanifold N in M are equivalent.

i) N is the image of an embedding of a k dimensional manifold.

ii) For all $p \in N$ there exists a chart (U, φ) such that $\varphi(U \cap N) = \varphi(U) \cap (\mathbb{R}^k \times \{0\})$.

Mandatory Exercise 4. (5 points)
Let $i : N \to M$ an embedding. Show that

$$(di)_p(T_pN) = \{v \in T_{i(p)}M | v(f) = 0 \quad \text{for all functions} \quad f : M \to \mathbb{R} \quad \text{such that} \quad f\big|_{i(N)} \quad \text{is constant}\}.$$

Suggested Exercise 1. Let X be a vector field on M. Show that the following are equivalent

1. X is smooth.
2. For every smooth function $f : M \to \mathbb{R}$ the function $X(f) : M \to \mathbb{R}$ is smooth.

Suggested Exercise 2. (0 points)
Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if} \quad t \leq 0 \\ e^{-1/t} & \quad\quad t > 0 \end{cases}$$

is smooth.

Suggested Exercise 3. (5 points)
Discuss which values of the maps below are regular.

a) Let M and N be manifolds and consider the projection $\pi : M \times N$ defined by $\pi(x, y) = x$.

b) Let $c \in \mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be the map $f(x) = |x - c|^2$.

c) The determinant $\det : M_2(\mathbb{R}) \to \mathbb{R}$ and trace $\operatorname{tr} : M_2(\mathbb{R}) \to \mathbb{R}$ of (2×2)-matrices $M_2(\mathbb{R})$.

Hand in: Monday 15th May
In the pigeonhole on the third floor of the MI